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Level I Stability Analysis (LISA) Documentation for Version 2.0

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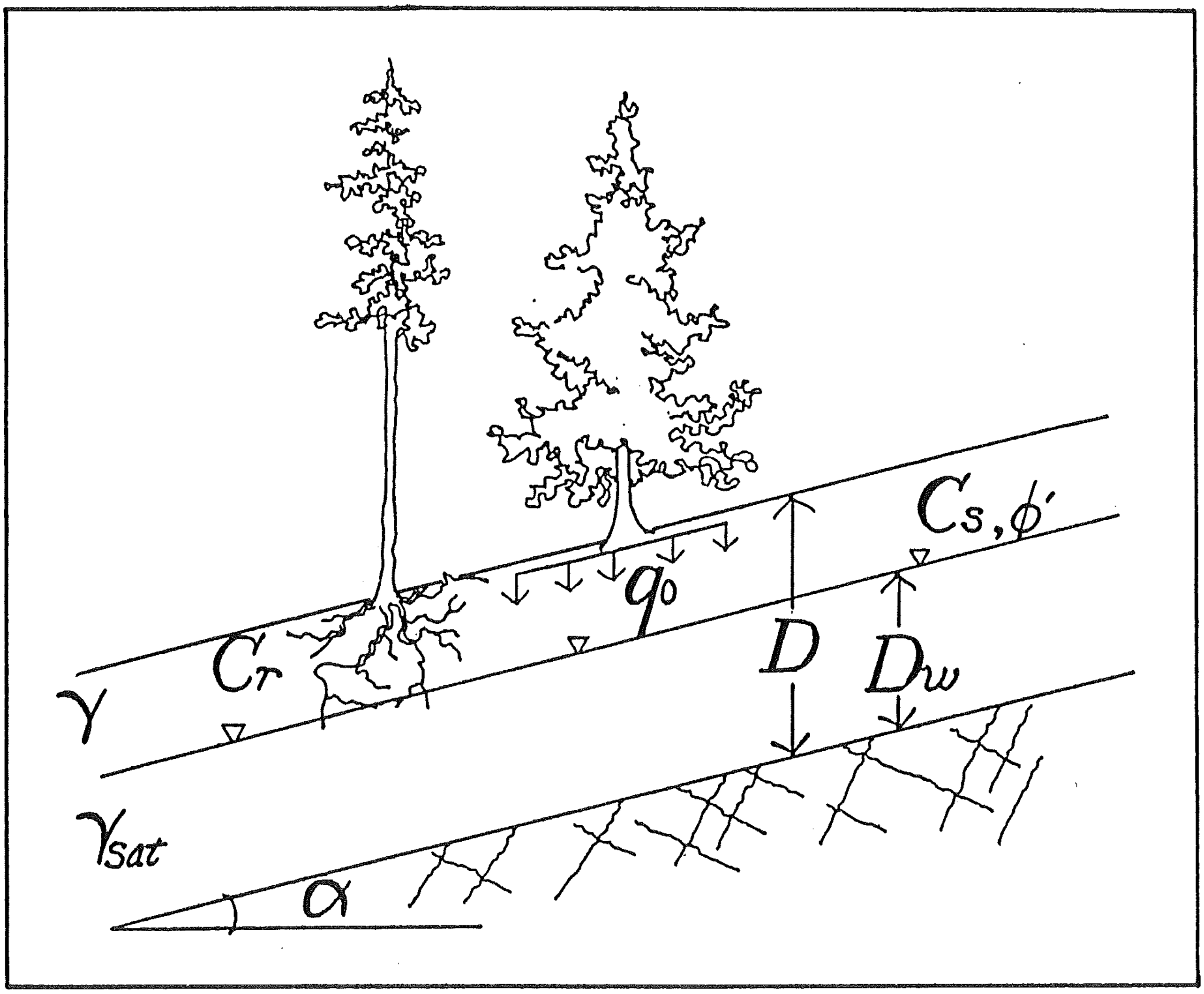

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Efforts have been made to see that LISA is reliable, but it is a model of reality, not reality itself. The user should have a thorough understanding of the model and should compare results to actual field conditions.

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ACKNOWLEDGMENTS

The development of LISA has been a team effort with many participants. Rod Prellwitz (Intermountain Research Station) recognized the need for and conceived the idea of a probabilistic Level I landslide hazard analysis using the infinite slope stability model. He played an integral role throughout the LISA program development. Gordon Booth (Intermountain Research Station) initially suggested the Monte Carlo approach to us. Dr. Terry Howard and Dr. Clarence Potratz (University of Idaho) and Dr. Henry Shovic (Gallatin National Forest) were involved in early LISA development. Several of our users, in particular the Gifford Pinchot National Forest geotechnical group, have contributed greatly by their efforts in applying LISA to field situations and effectively communicating the results to land managers. Their feedback to us during the development process has been enormously helpful and is much appreciated. We also appreciate the drafting of figure 5.7 by Karl Anderson, Gifford Pinchot National Forest.

A CAVEAT

LISA is a *tool* to be used by investigators who have some knowledge and experience concerning landslide behavior and geotechnical properties of soils. It requires engineering judgment and common sense, both in developing input distributions and interpreting the results. It does not give a unique "right" answer. It is a tool to help the user understand slope stability processes, quantify observations and judgments, and document and communicate those observations and judgments to other geotechnical specialists and to land managers. Do not rely on LISA alone, but add it to your existing toolbox. Any answer that one desires can be obtained by altering the input data. Without rationally justifying the input used, and without correctly understanding and interpreting the output, LISA becomes little more than a game of numbers.

Furthermore, LISA does not provide a complete risk analysis. The consequences of slope failures (such as the potential for damage to timber and fisheries resources, roads or structures, or the potential for injury or loss of life) should be assessed by users.

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RESEARCH SUMMARY

The Level I Stability Analysis (LISA) computer program is a *tool* to help estimate the relative stability of natural slopes or landforms. LISA results are intended to support management decisions at the multi-project or resource allocation level of planning. The primary use of the probability of failure estimated using LISA is to make qualitative, relative comparisons between the stability of landforms, and to identify areas that should be targeted for additional analysis. LISA also can be used to estimate the relative decrease in stability of a landform after timber harvest due to a potential reduction in estimated tree root strength and an increase in groundwater levels. The probability of failure also can be used quantitatively in a risk analysis, such as an expected monetary value (EMV) decision analysis.

LISA uses the infinite slope stability model to compute the factor of safety against failure for a given set of *in situ* conditions. The factor of safety (FS) is the ratio of the forces resisting a slope failure (tree root strength and soil shear strength) to the forces driving the failure (gravity). A slope with an FS greater than 1 is expected to be stable; a slope with an FS less than 1, unstable. The calculation of an FS with a single set of input values is called a *deterministic analysis*. However, it is recognized that there are variability in *in situ* conditions on a given slope or landform and many uncertainties in estimating input values for the variables in the infinite slope equation. Therefore, LISA uses Monte Carlo simulation to estimate the *probability* of slope failure rather than a single FS value. Monte Carlo simulation is useful for modeling an attribute that cannot be sampled or measured directly. The FS is such an attribute. A large number of Monte Carlo passes (say 1,000) is made with repeated random samplings of possible input values and the calculation of a factor of safety for each pass. The end result is a histogram of the calculated factors of safety and the probability of failure. LISA calculates the probability of failure by dividing the total number of passes into the number of factors of safety less than or equal to 1.

It is common to view the probability of an event as the likelihood of the event occurring. This meaning does not work well for the probability of failure in a large, variable landform. Viewing the probability of failure as the relative frequency of failure events is more realistic. For purposes of estimating the consequences of failure, the probability of failure also can be thought of as the portion of the land area in, or potentially in, a failed state during the period appropriate to the analysis. However, this meaning should be used with caution. The validity of the meaning depends on the scale of the analysis and should be checked with a landslide inventory. LISA does not simulate the actual number of failures, nor the size or location of individual failures. LISA provides the hazard, but the potential consequences still must be evaluated by the user.

CONTENTS

Level I Stability Analysis (LISA) Documentation for Version 2.0

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Introduction

This report provides comprehensive information required to use LISA intelligently. The goal was to give the user a sufficient reference manual for understanding and obtaining input distributions, understanding the concepts and methods LISA uses, and interpreting LISA results, and to be a guide for program operation. This report, therefore, has been divided into two parts: part 1 is the reference manual and part 2, the program operations guide. *Important points throughout this report are set in italic and marked by a block in the outside margin.*

It is *essential* that the user understands the fundamentals and concepts presented in part 1, or meaningless or misleading results might be obtained using LISA. However, understanding part 1 may be made easier for the uninitiated user if one first becomes familiar with LISA by running the demonstration exercise in part 2, chapter 3.

The Research Summary explains what LISA does and what the program results mean. It can be included, along with user additions, in reports of LISA results to help land managers understand the methods that have been applied.

The detailed Table of Contents functions as a reference device, assisting the reader in locating subjects by page number. The numeric system used to identify section and subsection headings assists the reader in locating cross-referenced sections. A list of symbols can be found after this introduction.

Part 1 of this report consists of six chapters and four appendices. Chapter 1 introduces the philosophy behind probabilistic slope stability analysis. Chapter 2 reviews probability and statistics fundamentals. Chapter 3 describes the infinite slope equation and its sensitivity to various input parameters. Chapter 4 describes details of the methods used in the LISA program and interpretation of results. Chapter 5 discusses how to select input distributions and values describing those distributions, both in general and for each variable in the infinite slope equation. Chapter 6 contains two examples of the range of uses to which LISA can be applied. References cited in part 1 are given after chapter 6.

Appendix A shows the derivation of the infinite slope equation with a phreatic surface parallel to the slope. Appendix B gives a detailed literature review of root strength. Appendix C discusses the rationale for selecting the suggested PDF's for root strength. Appendix D discusses using rain or rain-on-snow return periods with LISA probabilities of failure to arrive at an estimate of the likelihood of failure events occurring.

Part 2 contains four chapters and four appendices. Chapter 1 gives installation instructions. Chapter 2 gives general principles on how to run LISA. Chapters 1 and 2 generally will be all that are needed to get the user started. Chapter 3 describes in detail LISA operation, including screen prompts and error messages and a demonstration exercise. Chapter 4 describes use of DLISA, the deterministic version of LISA. The reference cited in part 2 is given after chapter 4.

Appendix A describes how to download LISA and DLISA from the Data General computer at Moscow, ID. Appendix B describes how to use the Software Reference Center on the Data General computer in the Washington, DC, office to obtain information on the latest program revision. Appendix C lists the equations used in DLISA. Appendix D is a list of error messages from both LISA and DLISA with cross references into chapters 3 and 4.

Documented source code (Hall and Kendall 1992) is available separately by special request made to the authors. Additional examples of LISA applications have been described by Hammond and others (1992) and Hammond and others (1988).

List of Symbols

A	Angular grain shape
A	Area of soil in a root count sample
a	Minimum value specified for a uniform, triangular, or beta PDF
a_i	Root cross-sectional area for the ith size class
A_R	Total cross-sectional area of all roots in a root count
b	Maximum value specified for a uniform or beta PDF, and apex of a triangular PDF; also width of slice in infinite slope derivation
$B[a, b, P, Q]$	Notation specifying a beta PDF
bf	Board foot = 12 by 12 by 1 inches
$BN[\bar{x}, s, r]$	Notation specifying a bivariate normal PDF
c	Maximum value specified for a triangular distribution
C'_a	Apparent soil cohesion caused by interpretation of a curved Mohr-Coulomb failure envelope
C_{app}	Apparent soil cohesion caused by capillary suction
CDF	Cumulative distribution function
CH	USC designation for fat clay
CL	USC designation for lean clay
c_v	Sample coefficient of variation $= s/\bar{x}$
C_v	Population coefficient of variation $= \sigma_X/\mu_X$
$cov[X, Y]$	Sample covariance between X and $Y = r s_X s_Y$
$Cov[X, Y]$	Population covariance between X and $Y = \rho \sigma_X \sigma_Y$
C_r	Additional shear strength caused by tree roots
C'_s	Effective soil cohesion
D	Soil depth measured vertically
D_{ac}	Apparent soil depth measured along a cutslope face

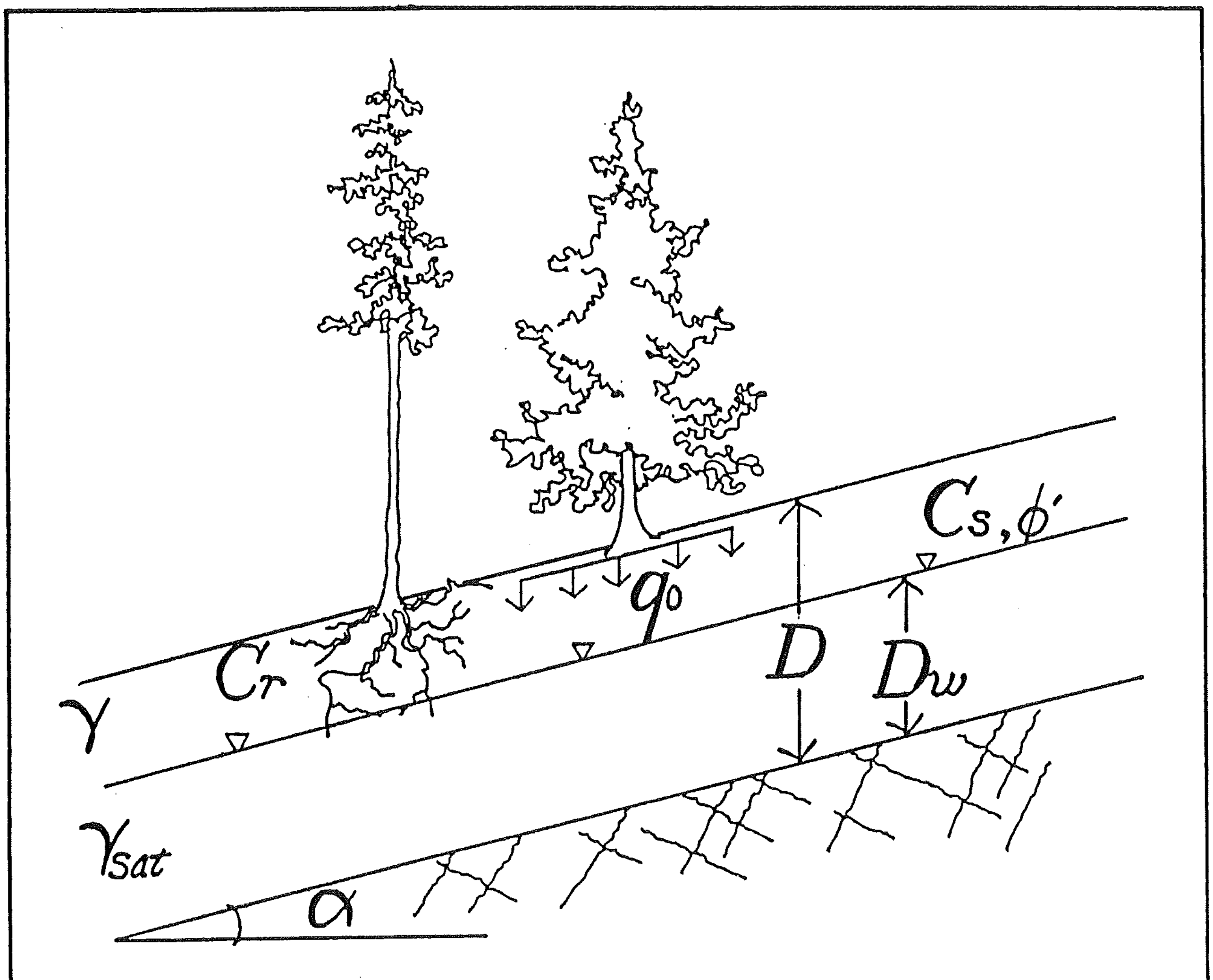
D_{as}	Soil depth measured using seismic refraction; depth is measured perpendicular to the refractor surface
D_b	Bulk density of minus 2mm fraction of soil
D_m	Vertical height of soil above the phreatic surface $= D - D_w$
D_r	Relative density of soil
D_R	Diameter of a root
D_w	Vertical height of phreatic surface
$\mathrm{E}[X]$	Expected value of a random variable X
$\mathrm{Var}[X]$	Variance of a random variable $X = \sigma_X^2$
E_R	Longitudinal stiffness modulus of a root
FS	Factor of safety
F_i	Average resisting tensile force of roots in the ith size class; also fraction of observations in the ith class of a histogram PDF
ft or '	Foot
$f(x)$	Function of x that describes the Y-ordinate of a PDF curve
GC	USC designation for clayey gravel
GM	USC designation for silty gravel
GP	USC designation for poorly graded gravel
GRC	Geologic resources and conditions data base
GRI	Geologic resource inventory
G_w	Specific gravity of solids
GW	USC designation for well-graded gravel
$H[k, f_1, \cdots, f_n]$	Notation for histogram PDF with k classes and f_n percent in each class
h_p	Vertical height of equipotential line
in or "	Inch
k	Number of classes in a histogram PDF
$k_{\phi 1}, k_{\phi 2}$	Empirical coefficients to estimate angle of internal friction from D_r
lb	Pound
LISA	Level I stability analysis
$L[x, s]$	Notation specifying a lognormal PDF
LSI	Land systems inventory
MH	USC designation for plastic silt
ML	USC designation for non plastic silt
N	Number of years

N	Normal force at the base of the slice in the infinite slope derivation
N'	Effective normal force
n	Number of data values or observations in a sample
$N[\bar{x}; s]$	Notation for a normal PDF with mean $\bar{x}$, and standard deviation s
n_i	Number of roots in size class i
OC	Organic carbon content of soil
OCR	Overconsolidation ratio
P	One of two shape parameters for a beta PDF
pcf	Pounds per cubic foot
PDF	Probability density function
PI	Plasticity index
psf	Pounds per square foot
$P[A \cap B]$	The occurrence of event B given that event A has occurred $= P[A]P[B \backslash A]$
$P[B]$	The probability that event B will occur
$P[\bar{B}]$	The probability that event B will not occur $= 1 - P[B]$
$P[B \backslash A]$	The probability that event B will occur given that event A has occurred. This is called a conditional probability and is used when the probability of event B depends on the occurrence of event A.
$P[FS \leq 1]$ or P_f	The probability of FS being less than or equal to 1, or the probability of slope failure
Q	One of two shape parameters for a beta PDF
q_0	Tree surcharge
r	Sample correlation coefficient
r^2	Coefficient of determination
R	Rounded grain shape
R_{pi}	Return period of event i
$r.v.$	Random variable
s	Standard deviation of sample data
S	Soil shear strength
SA	Subangular grain shape
SARA	Stability analysis for road access (Level II)
SC	USC designation for clayey sand
SM	USC designation for silty sand
SP	USC designation for poorly graded sand

SPT	Standard penetration test (ASTM D-1586)
SR	Subrounded grain shape
SRI	Soil resource inventory
s_X	Standard deviation of sample data for random variable X
T	Shear force acting on the base of a slice in the infinite slope derivation
T_i	Average tensile strength per foot cross-sectional area for the ith class
T_n	Normal component of root tensile resistance
T_r	Tensile strength of individual roots
T_R	Weighted average root tensile strength per average root cross-sectional area of all size classes
t_R	Average tensile root strength per unit soil area
T_s	Tangential component of root tensile resistance
$T[a, b, c]$	Notation for a triangular PDF
T-99	ASHTO test designation for the standard Proctor test
u or u_w	Pore-water pressure
u_a	Pore-air pressure
U	Uplift force on the base of a slice in the infinite slope derivation caused by pore-water pressure, u
USC	Unified Soil Classification system (ASTM D-2487-85 and D-2488-84)
$U[a, b]$	Notation for a uniform PDF
Var[X]	Variance of a random variable $X = \sigma_X^2$
w	Gravimetric moisture content = weight of water/weight of solids x 100%
W_T	Total weight of a soil slice in the infinite slope derivation
$\bar{x}$	Mean of sample data set
X	Random variable X
Y	Random variable Y
z	Thickness of the shear zone
α	Natural slope angle, in percentage or degrees in DLISA; in percentage in LISA
β	Artificial slope angle in degrees or ratio
γ or γ_m	Moist soil unit weight
γ_d	Dry soil unit weight
$\gamma_{\max}$	Maximum unit weight obtained with a standard laboratory test such as standard (T-99) or modified (T-180) Proctor test

γ_{sat}	Saturated soil unit weight
γ_w	Unit weight of water = 62.5 pcf
θ	Angle of root shear distortion
μ_l	Mean of the logarithm of data values
μ_X or μ	Mean of random variable X
ρ	Population correlation coefficient
σ' or σ'_n	Effective normal stress
σ_l	Standard deviation of the logarithm of data values
σ_R	Tensile stress developed in the root at the shear plane
σ_X or σ	Standard deviation of random variable X
τ	Soil shear strength; also shear stress
τ_R	Skin friction stress along a root
ϕ'	Effective angle of internal friction
ϕ'_a	Apparent effective angle of internal friction caused by interpretation of a curved Mohr-Coulomb failure envelope
ϕ^b	Slope of the line relating capillary suction and apparent soil cohesion
ϕ'_r	Residual angle of internal friction
ϕ'_p	Peak angle of internal friction
ϕ'_{ult}	Ultimate angle of internal friction, equivalent to ϕ'_r
ϕ_μ	Particle-to-particle friction angle
$\hat{}$	Estimation of a population (true) parameter from sample data (e.g., $\hat{\mu}$ or $\hat{\sigma}$)

Part 1—Reference Manual



CHAPTER 1—CONCEPTS IN PROBABILISTIC SLOPE STABILITY ANALYSIS

1.1 Applicability of LISA

LISA is a probabilistic model intended to be used primarily for relative landslide hazard evaluation for resource allocation, forest planning (land management plans), timber sale allocation, environmental assessment reports (EARs), and transportation planning (Prellwitz and others 1983). LISA can delineate areas susceptible to broad-scale landslides to alert land managers as to where the hazard is greatest. LISA also can be useful for project planning (Level II) to evaluate the stability of natural slopes in cutting units and the effects of timber harvest on stability.

LISA is a *tool* to be used by investigators who have some knowledge and experience concerning landslide behavior and geotechnical properties of soils. It requires engineering judgment and common sense, both in developing input distributions and interpreting the results. It does not give a unique "right" answer. It is a tool to help the user understand slope stability processes, quantify observations and judgments, and document and communicate those observations and judgments to other geotechnical specialists and to land managers. Do not rely on LISA alone, but add it to your existing toolbox. Any answer that one may desire can be obtained by altering the input data. Without rationally justifying the input used, and without correctly understanding and interpreting the output, LISA becomes little more than a game of numbers.

LISA does *not* provide a complete risk analysis; the impact or consequence of potential failures needs to be evaluated by the user. For example, the user may want to assess the potential for damage to timber and fisheries resources or to roads or structures, or the potential for injury or loss of life resulting from slope failures.

1.2 What Is a Probabilistic Analysis? (Deterministic vs. Probabilistic Analysis)

Typically in day-to-day engineering work, slope stability analyses are performed using limit equilibrium equations to obtain a calculated factor of safety against failure. A slope with a factor of safety greater than 1 is expected to be stable, whereas a slope with a factor of safety less than or equal to 1 is expected to be unstable. This calculation of a single factor of safety, given a single set of input values, is a deterministic analysis. However, it is recognized that there are many uncertainties in estimating input values for an analysis. Variability and uncertainty in soil shear strength parameters are due both to variation in soil properties across the site and to measurement errors in field and laboratory testing. Groundwater levels vary spatially and temporally. There are uncertainty and variability in the other factors as well, all of which yield uncertainty as to the precise meaning of the factor of safety value. That is, it is recognized that a slope with a calculated factor of safety of 0.9 may not fail, and one with a calculated factor of safety of 1.1 might fail. Thus, design factors of safety of 1.2 to 1.5 often are used to give the engineer a conservative buffer against uncertainty and spatial variability.

A probabilistic analysis provides an estimate of the probability of slope failure, rather than the factor of safety, by using probabilistic models to quantify the uncertainty and variability associated with the prediction of slope stability. The primary advantage of a probabilistic analysis is that it logically and systematically accounts for uncertainty and variability in the stability analysis and communicates to all concerned that uncertainty and variability have been considered. With a probabilistic analysis, a single value for each input parameter is no longer required. Rather than modeling a site as homogeneous, we can deal with the site's variable factors.

A probabilistic analysis also provides results that can serve as input for a decision-making analysis in the light of recognized uncertainty. Such analyses require a probability of failure (in other words, hazard) and the consequences of failure in order to evaluate risk. In a risk analysis, the hazard and its consequences associated with various decision alternatives are evaluated to aid in decision making. In the context of the following discussion, *hazard* is defined as the calculated probability of slope failure, and *risk* is defined as a measure of the socioeconomic consequences of slope failure (susceptibility to losses). Two slopes may have the same estimated probability of slope failure and therefore the same hazard (as estimated by LISA). However, if a bridge or an anadromous fisheries stream lies below one of the slopes and not the other, the risks associated with failure of the first slope are much greater than are those associated with the other slope. Comprehensive risk analysis is beyond the scope of this manual.

1.3 How to Perform a Probabilistic Analysis–Monte Carlo Simulation

Most probabilistic methods described in the literature focus on the analysis of individual slopes and consider only the variability of soil cohesion, angle of internal friction, or groundwater, or a combination of these. A closed-form solution is derived for the mean and standard deviation of the factor of safety, which has an assumed probability distribution (usually normal, lognormal, or beta), and then a probability of failure is calculated (Chowdhury and Tang 1987). One problem with these methods is that the variabilities of other important factors, such as slope and soil depth, are not considered. One reason all factors are not considered as stochastic variables is that the calculus needed to evaluate the integrals resulting from the derivation of the probability distribution of the factors of safety would not be tractable. However, when analyzing large areas, as in resource planning, all of the input factors have sufficient spatial variability and measurement uncertainty to warrant treatment as stochastic variables.

An alternative method used to evaluate landslide hazard is Monte Carlo simulation. Monte Carlo simulation is useful for modeling an attribute that cannot be sampled or measured directly but can be expressed as a mathematical function of properties that can be sampled or described. Factor of safety is such an attribute. Monte Carlo simulation is the method used in LISA because of its capability to incorporate the variabilities of many input parameters, as is required for a stability analysis of large, variable landforms using the infinite slope model.

If we want to predict a possible value of the factor of safety, we take a possible value for each input variable and use the appropriate performance function (a stability equation) to calculate the corresponding value of the factor of safety. This is known as one Monte Carlo pass or iteration. In Monte Carlo simulation we generate a large number of factor of safety values (say 1,000) by repeated, random, independent samplings of a set of possible input values and calculate a corresponding factor of safety value for each pass. The set of possible input values for each input parameter is described by a probability distribution. The final simulation output is a set of 1,000 possible factor of safety values that can be displayed as a histogram. The relative frequencies of these 1,000 values are assumed to be representative of the frequencies we would have obtained had we analyzed all possible combinations of the input variables. Thus, the relative frequency of the computed factors of safety less than or equal to 1 is an estimate of the probability of occurrence of factors of safety less than or equal to 1 in nature (as defined by the user). We obtain the probability of failure by dividing the total number of passes into the number of calculated factor of safety values less than or equal to 1.

1.4 Meaning of the Probability of Failure Estimated by LISA

The probability of failure, strictly speaking, is the total number of Monte Carlo iterations divided into the number of calculated factors of safety with a value less than or equal to one. In other words, it is the relative frequency with which possible values of the factor of safety in the simulation are less than or equal to one. The probability of failure estimated by LISA should be reported as a *conditional* probability given that the considered storm event with the resulting groundwater distribution used in the analysis occurs.

It is common to view the probability of an event as the likelihood of that event occurring. This meaning does not work well for the probability of failure in a large, variable landform, because the possibility of just one failure occurring in the landform gives a probability of landslide occurrence of one. It is more useful to think of the probability of failure of a large landform as the relative frequency of failure events. For example, if landform A has a probability of failure of 0.05 and landform B has one of 0.025, we would expect landslides to be twice as severe, in number or size, in landform A. The probability of failure can be viewed as the probability of landslide occurrence *if* the area analyzed is small enough (i.e., one slope or one drainage) so that only one failure could occur within the area.

With few data, the input distributions represent one's uncertainty about the variables as well as one's best guess about their spatial variability across the landform. Therefore, because of the two-dimensional nature of the infinite slope analysis, the estimated probability of failure can best be thought of as the likelihood that any possible randomly selected cross-section through the slope would be analyzed as unstable. As more data are available, the probability distribution of each input variable represents more the spatial variability of that variable and less the uncertainty. Here the probability of failure should be an estimate of the expected percentage area of the landform involved in failure *during the period appropriate to the analysis*, that is, during the period of minimum root strength following timber harvest, or during the rain or snowmelt event causing the groundwater levels used in the analysis.[1] Thinking of the probability of failure as the expected portion of the landform in, or potentially in, failure can indicate to management the magnitude of consequences to expect. However, this meaning for the probability of failure should be verified by comparison with field observations.

Landslide inventories provide the best means to verify whether the estimated probability of failure values are reasonable. Landslide inventories traditionally have been used to assess relative hazard by drawing the inductive conclusion that landslides will occur again in areas where they have occurred previously.

[1] A method for conditioning the LISA probability of failure estimates with the probability of certain rainfall or snowmelt events occurring during some specified length of time is discussed in appendix D.

Therefore, areas with many inventoried landslides should have a high probability of failure as predicted by LISA. When considering the percentage of land area involved in landslides, we must realize that these "high" probabilities of failure may actually occur on a small portion of the landscape. Published landslide inventories report values on the order of 0.5 to 15 percent of the area inventoried (Ice 1985). If the input distributions are based on subjective estimates rather than estimated from actual data, there may be only a relative comparison between probabilities of failure predicted by LISA and percentage of area in slope failure. But we still should see the relative relationship that areas with a higher probability of failure as predicted by LISA should have a higher frequency of occurrence of landslides than do areas with a lower probability of failure.

As with any computer program, "garbage in = garbage out." *If the input distributions do not describe realistically the values and distributions of the factors on the ground, then the simulated probability of failure will not provide a realistic measure of landslide hazard.* A method for quantifying the reliability in the LISA results is desirable based on whether the initial input distributions are formulated from field measurements or from subjective estimates. With such a method, as more field measurements are made and subsequent data are fed back into LISA from Level II and Level III field investigations, the increase in reliability of the LISA simulation can be documented. Methods to accomplish this currently are under study.

1.5 Use of the Probability of Failure

The probability of failure can be used qualitatively to make relative comparisons between landforms to identify areas that should be targeted for additional analysis. The probability of failure also can be used quantitatively in a risk analysis, such as an expected monetary value (EMV) decision analysis. Research efforts are continuing in this area.

Often in land management planning, one has to make subjective judgments about what probability of failure is acceptable. *Interpretation of the probability of failure as the percentage area expected in failure can help geotechnical specialists recommend to land managers what probability of failure is excessive. However, the possible consequences of failure, such as an estimate of the quantity of material that may impact downslope lands or streams, also need to be addressed.*

Reporting a single probability of failure value tends to imply precision in the results. Therefore, *we encourage users to report a range of probability of failure values obtained from several simulations using the same input distributions* but different random number seeds. Also, one may perform sensitivity analyses with LISA, changing the shape and values describing the input distributions over realistic ranges to see how the probability of failure is affected. Again, the range of values obtained should then be reported. Used as an iterative tool, LISA can help the user document personal judgments and observations about an area, communicate them to land managers and to other geotechnical specialists, and help identify factors critical to landslide hazard assessment in a given area.

1.6 Limitations of the LISA Analysis (What LISA Does Not Do)

LISA does not simulate the size or number of failures that might occur on a particular landform. Nor can LISA predict exact locations of any failures, or the type of failure (although it should give more accurate results for translational failure modes). Therefore, LISA cannot be used to directly estimate the consequences of failure, such as whether sediment will reach a stream, or the volume of sediment delivered.

1.7 How Factor of Safety Relates to Probability of Failure

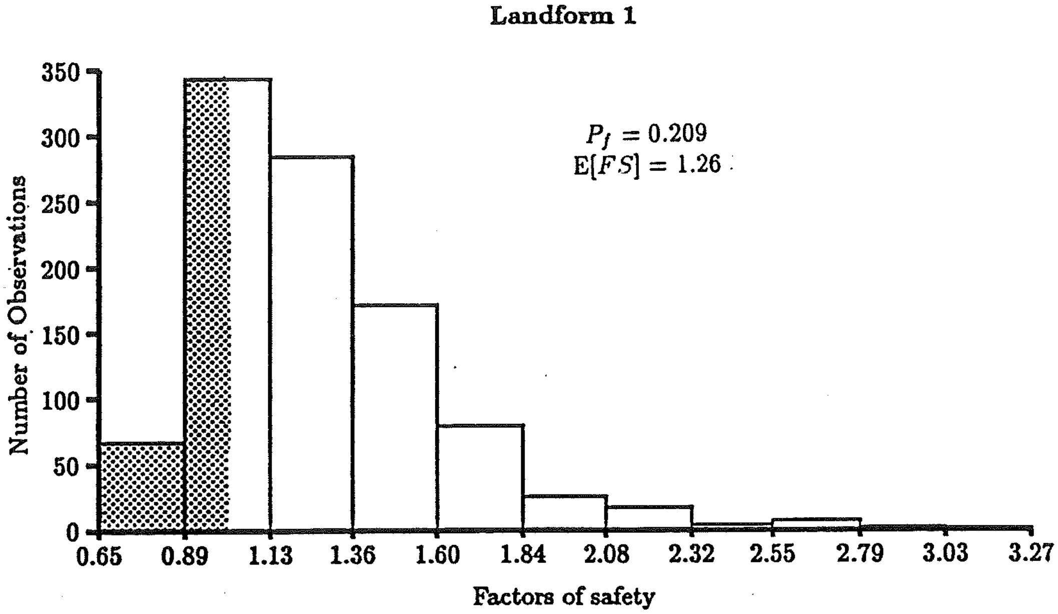
One approach to estimating a "likelihood" of failure is to measure or estimate either average or conservative values for each variable, and calculate a factor of safety deterministically. If the resulting factor of safety is fairly high, say 1.2, one could conclude that the likelihood of failure would be low. But how low depends on whether average or conservative input values were used, and what the possible variation in factors of safety is. In this section, we will discuss three concepts concerning the relationship of factor of safety to probability of failure.

The first concept is that the mean factor of safety for a landform is not a good indication of the probability of failure. This is because the probability of failure depends not only on the mean, but also on the variance of the factors of safety, which is controlled by the variance in the input distributions. An example is given in figure 1.1 and table 1.1 in which LISA gave similar mean factors of safety (1.26 and 1.19) for two hypothetical landforms, but a much higher probability of failure for landform 1, which has larger standard deviations for the input distributions. Table 1.1 shows the input distributions used. One should be aware that larger standard deviations for the input distributions might lower the probability of failure when the mean factor of safety is less than one.

The second concept is that the deterministic factor of safety calculated from the mean values of each input distribution may not equal the mean of the distribution of the factor of safety values, even when all of the input distributions are symmetrical. Take, for example, landform 1 in table 1.1. The mean values of the input distributions yield a deterministic factor of safety of 1.18 while the mean of the distribution of factors of safety from Monte Carlo simulation is 1.26. This is due to the fact that the factor of safety distribution for landform 1 is skewed to the right, which shifts the mean factor of safety to a higher value than that for a symmetrical distribution. In general, the expectation (or mean) of a nonlinear function, in our case the infinite slope equation, is not equal to the value of the function obtained when the mean values of each input variable are used in the function. Therefore, the mean of the factor of safety distribution should not necessarily be used as a substitute for a deterministic value (or vice versa), particularly when the distribution is highly skewed. The mean is just one measure of central tendency of the distribution. Commonly, the median or the mode value is closer to the deterministic factor of safety value than is the mean value.

Note that the mean value of the distribution for landform 2 (1.19) is very close to the value obtained from the means of the input variables (1.18). This happens because the distribution of factors of safety for landform 2 is relatively symmetrical.

The third concept, often difficult for engineers to comprehend when first introduced to probabilistic concepts, is that a slope with a computed factor of safety of 1.0 is not necessarily on the verge of failure. The probability of failure is not 1.0. In fact, the probability of failure is on the order of about 0.4 to 0.6,



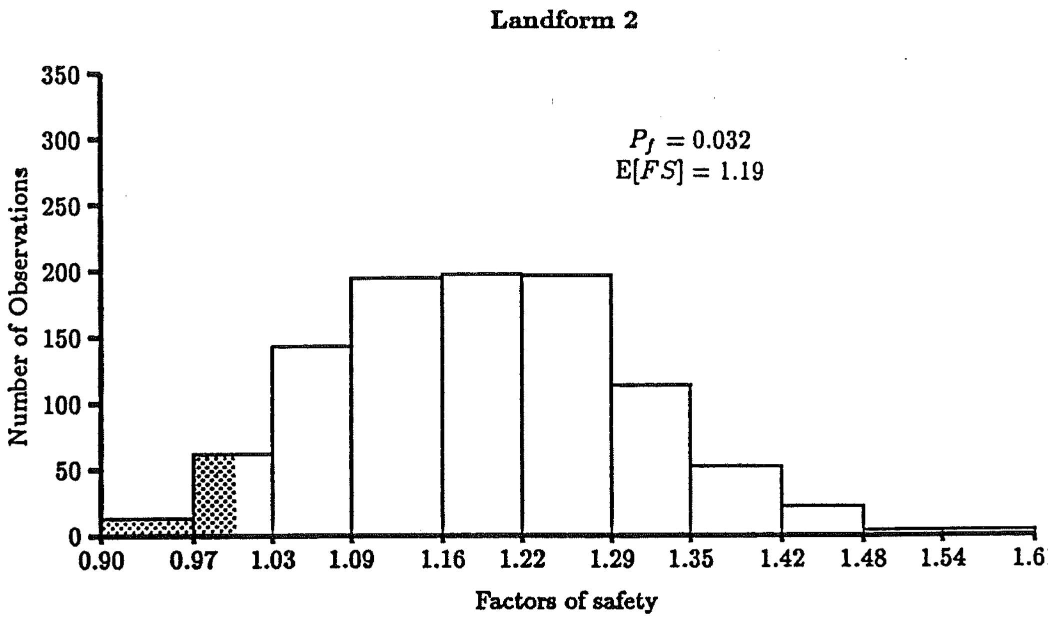


Figure 1.1—The distributions of factor of safety for two landforms. The landforms have nearly the same mean factor of safety but quite different probabilities of failure. The shaded area in each histogram represents values of factor of safety less than 1.

depending on the skewness of the factor of safety distribution. If the factor of safety distribution is symmetrical, the probability of failure is 0.5. This illustrates that the computed factor of safety may not be a good predictor of the behavior of the slope because of the natural variability of the physical factors and because of our inability to know without error the values of these factors.

Table 1.1—Distributions used for figure 1.1

	Landform 1			Landform 2		
	Distribution	$\bar{x}$	s	Distribution	$\bar{x}$	s
Soil depth	$T[1,4,7]$	4.0	1.2	$T[3,4,5]$	4.0	0.4
Slope	$U[60,80]$	70.0	5.8	$T[65,70,75]$	70.0	2.0
Tree surcharge	$U[5,15]$	10.0	2.9	$U[5,15]$	10.0	2.9
Root cohesion	$U[20,140]$	80.0	35.0	$T[50,70,120]$	80.0	14.6
Friction angle	$N[34,1]$	34.0	1.0	$N[34,0.5]$	34.0	0.5
Soil cohesion	$N[50,15]$	50.0	14.6	$N[50,10]$	50.0	10.0
Dry unit weight	$N[100,1]$	100.0	1.0	$N[100,1]$	100.0	1.0
Moisture content	$N[20,0.5]$	20.0	0.5	$N[20,0.5]$	20.0	0.5
Specific gravity	2.66			2.66		
D_w/D	$U[0.4,1]$	0.7	5.8	$T[.5,.7,.9]$	0.7	0.08
Factor of safety	see fig. 1.1	1.26	0.3	see fig. 1.1	1.19	0.1
Deterministic factor of safety	1.18			1.18		

CHAPTER 2—PROBABILITY THEORY REVIEW

The user should be familiar with the following concepts and terms when using LISA. We advise that you also read a good textbook if the material is new to you. Readable discussions are contained in Benjamin and Cornell (1970), Newendorp (1975), and Smith (1986).

2.1 Definitions and Relationships

EVENT

In probability theory, it is assumed that a random experiment, or sampling exercise, will have outcomes that depend on chance. A collection of one or more outcomes is known as an *event.* For example, consider a laboratory testing program wherein the dry unit weight is determined for each of 10 soil specimens randomly selected from a Shelby tube sample. An outcome is one test result (say, 103 pcf). An event is a collection of outcomes, such as all test results greater than 106 pcf, or all results between 100 and 110 pcf.

$P[B]$ is the probability that event B will occur. $P[B]$ is a real number between 0 and 1 assigned to event B.

$P[\bar{B}]$ is the probability that event B will not occur and is known as the complement of $P[B]$. $P[\bar{B}] = 1 - P[B]$.

RANDOM VARIABLE

A *random variable* (*r.v.*) is a variable or attribute (such as a physical property or characteristic) that takes on different values according to the outcomes of repeated experiments or sampling events.

These values cannot be predicted with certainty; thus, each possible range of values has an associated probability (or likelihood) of occurrence. For this reason, *r.v.*'s often are called *stochastic* variables to indicate the stochastic, or probabilistic, nature of their values. The term *random* here does not imply that the variable itself is random or has randomly distributed values, but rather that the values occur in a probabilistic manner. In the previous example for *event*, the dry unit weight of the soil is an *r.v.* If the value of a variable is known with certainty or with negligible uncertainty (at the time of analysis or decision making), then the variable is called a *deterministic* variable.

PROBABILITY DISTRIBUTION

A *probability distribution* is a discrete or continuous function that defines the likelihood, or the probability, that a random variable will have some particular range of values. Probability distributions can be expressed in two forms: the cumulative distribution function (CDF) and the probability density function (PDF). These are shown in figure 2.1 and described below.

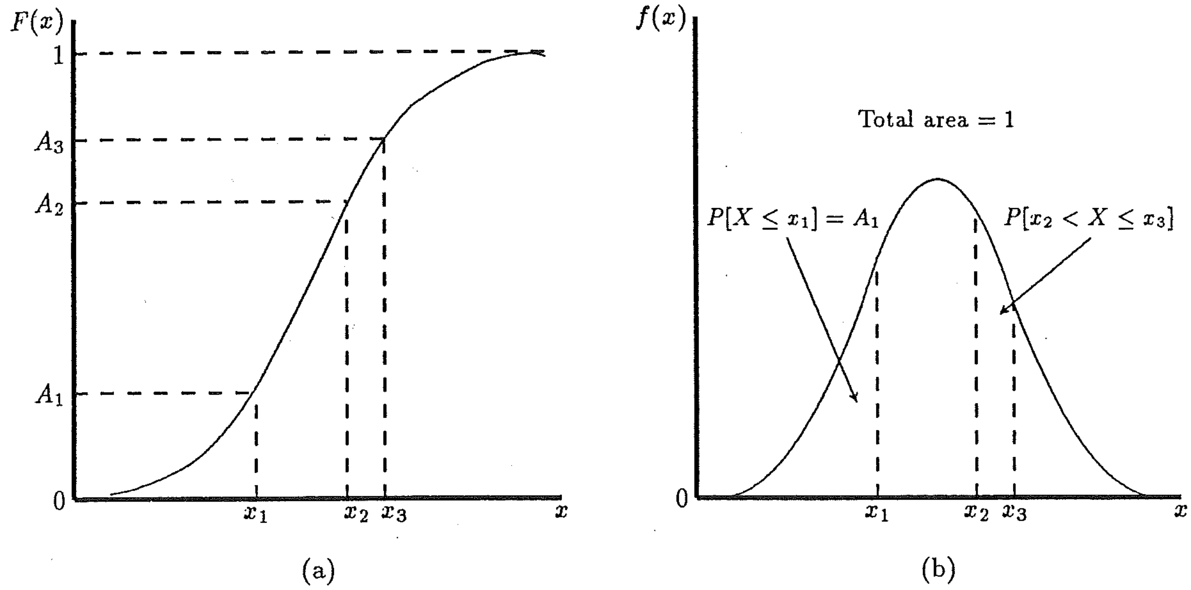


Figure 2.1—Example CDF (a) and PDF (b). In each, the probability that the random variable X takes on a value less than or equal to x_1 is equal to A_1. This is expressed mathematically as $P[X \leq x_1] = A_1$. The probability that the random variable X takes on a value between x_2 and x_3 is equal to $A_3 - A_2$ on the CDF, and to the area under the curve between x_2 and x_3 on the PDF.

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

The CDF for the *r.v.* X is a function that describes the probability that the *r.v.* X takes on a value less than or equal to x:

$$F(x) = P[X \leq x]$$

The properties of a CDF are:

- It has values between 0 and 1 inclusive.
- It is a nonnegative, nondecreasing function of a real-valued variable. A CDF can be defined for discrete or continuous *r.v.*'s.

PROBABILITY DENSITY FUNCTION (PDF)

The PDF for a continuous *r.v.* X is defined as:

$$f(x) = \frac{d[F(x)]}{dx}.$$

The properties of a PDF are:

- It is a nonnegative function where $\int_{-\infty}^{+\infty} f(x)dx = 1$
- The probability that the *r.v.* X will take on a value between x_2 and x_3 is equal to the area under the PDF curve between x_2 and x_3:

$$P[x_2 < X \leq x_3] = \int_{x_2}^{x_3} f(x)dx$$

This is illustrated in figure 2.1b. PDF's are used in LISA to describe input variability.

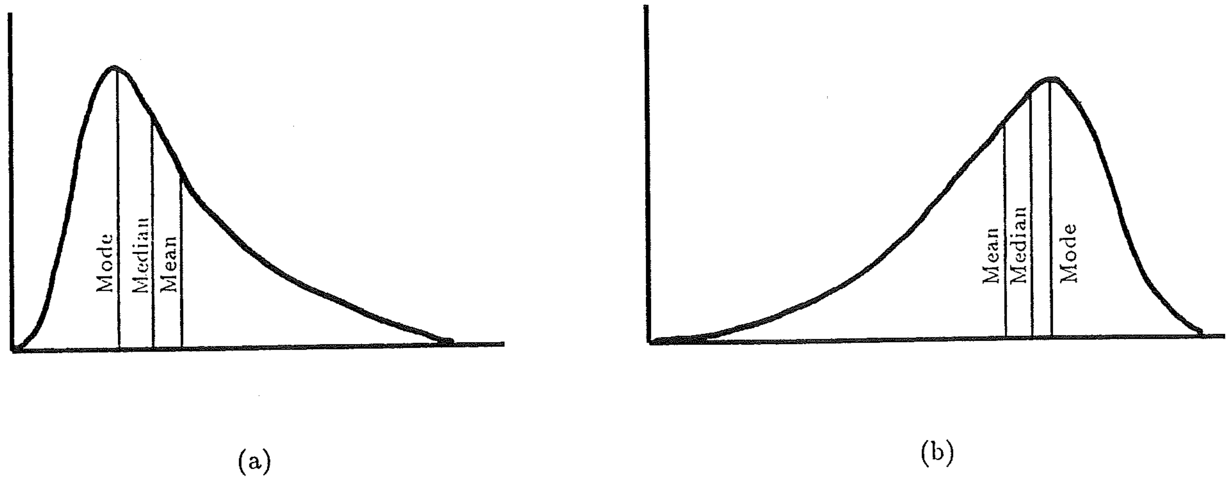


Figure 2.2—The relationship between mean, mode and median for a skewed right PDF (a) and a skewed left PDF (b).

Measures of Central Tendency— There are specific values that give important and useful information about a PDF. These values describe the central tendency of a PDF and the variability or range within which an *r.v.* is distributed. There are three measures of the central tendency of a PDF—the *mean*, the *median*, and the *mode*.

MEAN

The *mean* value of a PDF is the weighted average value of an *r.v.* where the weighting factors are the probabilities of occurrence. The mean value of a PDF is also called the expectation of the *r.v.* (E[X]). If the *r.v.* X has a known PDF (described by $f(x)$), then E[X] can be computed by:[2]

$$\mu_X = \mathrm{E}[X] = \int_{\text{all } x} x f(x) dx.$$

E[X] can be thought of and is mathematically equivalent to the centroidal axis of the PDF.

MODE

The *mode* of a distribution is the value that occurs with the greatest frequency, or the value that is most probable. Thus, it is the peak of the PDF curve. A distribution may have one mode, more than one mode, or no mode. A distribution having only one mode is called *unimodal.*

MEDIAN

The *median* of a distribution is the value of the *r.v.* corresponding to a vertical line that divides the PDF into two parts having equal areas. That is, there is a 0.50 probability that the *r.v.* will take on a value greater than (or less than) the median value.

[2]The general definition for expectation is: $\mathrm{E}[h(x)] = \int_{\text{all } x} h(x)f(x)dx$ where $h(x)$ is a function of x. The mean is a special case in which $h(x) = x$.

The mean, mode, and median all coincide for symmetrical PDF's. However, for asymmetrical PDF's, this will not be the case. Figures 2.2a and b illustrate the relationship between the mean, mode, and median for a distribution skewed to the right and a distribution skewed to the left, respectively. You should note that for the skewed distribution, the mean value is not the most probable value—the mode is. Often in deterministic studies, we think of the single value estimate as being the mean or average value. However, the mean is just one measure of central tendency of the distribution and may not necessarily be the best single value to use to characterize the distribution; the median or mode may be better (see also section 1.7 for additional discussion).

Measures of Variability— There are also three measures of variability. They are the *range* (the difference between maximum and minimum value), the *variance*, and the *standard deviation*.

VARIANCE

A common measure of the dispersion of the distribution of the *r.v.* X about its mean is given by the *variance* of X:

$$\mathrm{Var}[X] = \int_{\text{all } x} (x - \mu_X)^2 f(x)dx$$

If the variance is low, the values will be concentrated near the mean. If the variance is high, the values will be scattered over a wide range.

STANDARD DEVIATION

The *standard deviation* measures how far a typical or average value of the *r.v.* X deviates from the mean. It is computed as the positive square root of the variance of X:

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

The units on the standard deviation are the same as the units on the *r.v.*

COVARIANCE

The *covariance* between two random variables X and Y is a measure of the stochastic dependence between X and Y. It is defined as:

$$\mathrm{Cov}[X,Y] = \mathrm{E}[(X - \mu_X)(Y - \mu_Y)]$$

JOINT PROBABILITY DENSITY FUNCTION

When two random variables are being considered simultaneously, their joint behavior is described by a *joint probability density function.* Joint behavior need only be considered for LISA when the behavior of one random variable is dependent on another (for example, Cs' and ϕ', as discussed in section 4.2). A joint PDF is denoted by $f_{X,Y}(x,y)$.

MARGINAL PDF

A *marginal* PDF describes the relative likelihood of values of one of the variables considered in a joint PDF, irrespective of the other. A marginal PDF is denoted by $f_X(x)$.

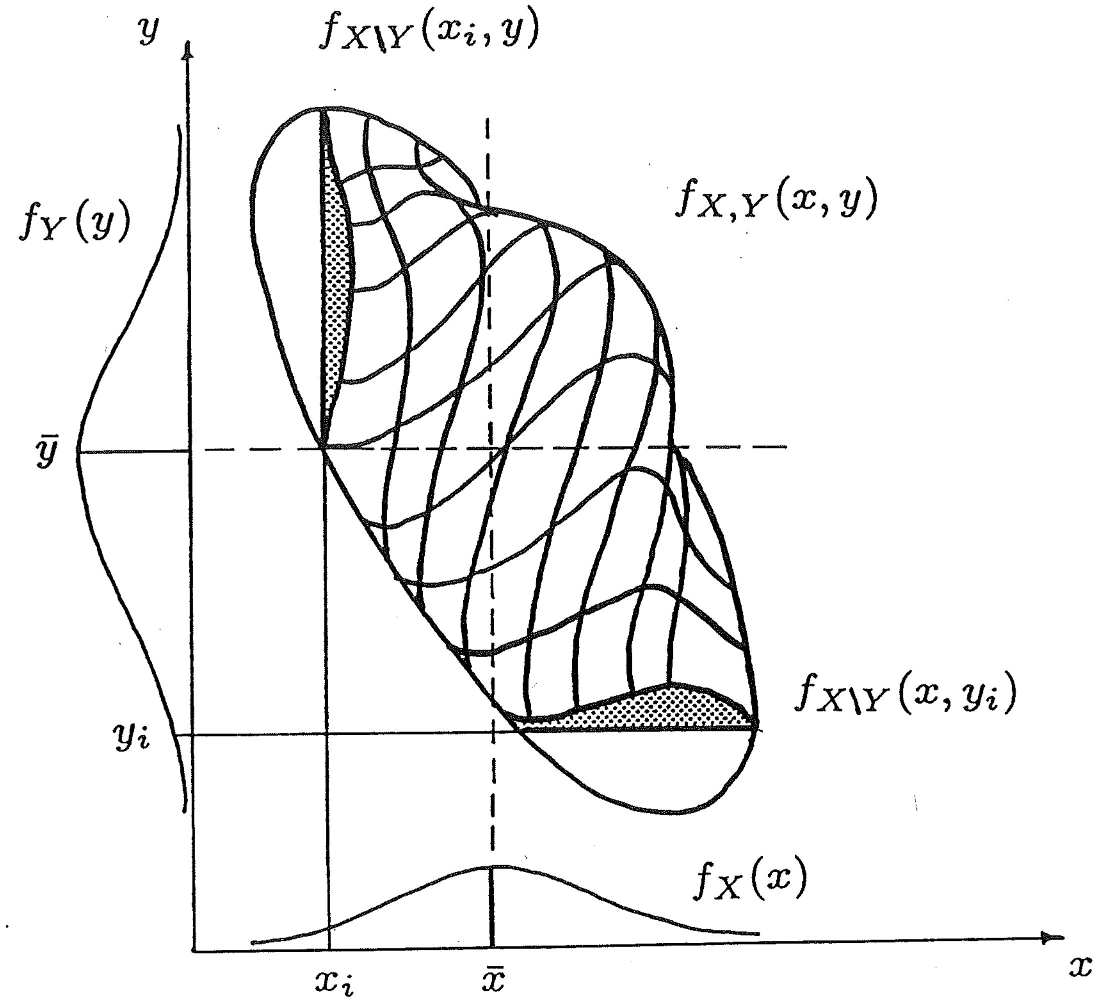


Figure 2.3—Joint PDF ($f_{X,Y}(x,y)$) illustrating a negative correlation between two random variables. Also shown are the marginal PDF's ($f_Y(y)$ and $f_X(x)$) and the conditional PDF's at y_i ($f_{X\backslash Y}(x, y_i)$) and x_i ($f_{X\backslash Y}(x_i, y)$). Note that the shaded areas shown as conditional PDF's technically are not the true conditional PDF's because the area under each curve does not equal 1. To be true conditional PDF's, they need to be normalized by dividing by $f_X(x)$ or $f_Y(y)$. However, the shaded areas graphically represent the conditional PDF's.

CONDITIONAL PDF

A *conditional* PDF describes the relative likelihood of values of one variable when one value of the other variable is given. A conditional PDF is denoted by $f_{X\backslash Y}(x, y_i)$.

The joint, marginal, and conditional PDF's are illustrated in figure 2.3.

MEAN AND STANDARD DEVIATION OF A STATISTICAL SAMPLE

In civil engineering and geology, the term *sample* means a single item, such as a soil sample. In statistics, the term *sample* means a set of items, test results, or values. To distinguish between the two meanings, the term *specimen* is preferred for an engineering or geological sample. Thus, we can speak of a sample of 20 soil specimens, or a sample of 25 slope measurements.

Generally, n specimens or measurements will yield one statistical sample. The mean of the sample, $\bar{x}$, can be calculated by:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

where n is the number of data values. The standard deviation of the sample can be calculated by:[3]

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}.$$

In LISA, as in most situations, the sample mean and standard deviation are used to estimate the population (true) mean and standard deviation when there are at least 30 data values. Thus:

$$\bar{x} = \hat{\mu}_X, \text{ where } \mu_X = \mathrm{E}[X]$$

$$s = \hat{\sigma}_X, \text{ where } \sigma_X = \sqrt{\mathrm{Var}[X]}$$

The $\hat{}$ denotes an estimated value.

COEFFICIENT OF VARIATION

The sample *coefficient of variation* (c_v) is a dimensionless measure of dispersion and is equal to the ratio of the sample standard deviation to the sample mean: $c_v = s/\bar{x}$.

CORRELATION COEFFICIENT

The *correlation coefficient* (r) is a measure of the *linear* dependence between two random variables. The value of r varies between -1 and $+1$. A negative sign ($-$) means a negative linear correlation, and a positive sign ($+$) means a positive linear correlation between the two *r.v.*'s. The correlation coefficient is defined as:

$$r = \frac{\mathrm{cov}[x, y]}{s_X s_Y}$$

If the *r.v.*'s X and Y are statistically independent, then their covariance, and r, are zero. However, r can be small even if their covariance is not small, such as in the case where X and Y are *nonlinearly* related. In addition, a high value of r can result for independent variables if, in a scatter plot, all of the values except one are clustered together, and the one outlier value lies well outside all the others. This is known as a spurious correlation. Therefore, it is highly recommended that you view a scatter plot of the data to ensure correct interpretation of the r value.

[3] Division by $n-1$ instead of by n is required here because s is obtained using one calculated term ($\bar{x}$), as well as using all of the data values. Thus, one degree of freedom is lost from the data set. In other words, if you were provided with 19 data values and $\bar{x}$ for a sample with 20 observations, you could calculate the 20th value (using $\sum(x_i - \bar{x}) = 0$). Thus, only $n - 1$ of the data values are freely determined, and the nth value depends on the others.

The correlation coefficient can be calculated by taking the square root of the coefficient of determination (r^2). The value of r^2 is a number between 0 and 1 (inclusive) that describes the fraction of the variation in Y that is explained by the variation in X, and can be obtained from a least-squares linear regression between *r.v.*'s X and Y. The sign of r is the same as the sign of the slope of the line obtained from the regression.

Important Relationships—For a constant c and a random variable X, $\mathrm{E}[cX] = c\mathrm{E}[X]$; $\mathrm{Var}[X + c] = \mathrm{Var}[X]$; and $\mathrm{Var}[cX] = c^2\mathrm{Var}[X]$. For two random variables X and Y, $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$.

2.2 Probability Distributions

The probability density functions (PDF's) used in LISA are described in this section. Formulas for the function ($f(x)$), the mean ($\bar{x}$), and the standard deviation (s) of each distribution are given in the figures. These formulas are for user reference only; the parameters that LISA requires for each PDF are shown as USER INPUT. A shorthand notation for each distribution that is used in this manual is also shown. Note that the y-axis of the PDF curve is labeled $f(x)$; that is, for a given value of the random variable X, the y-ordinate of the PDF curve will be given by the function $f(x)$. This function gives values for the y-axis such that the area under the PDF curve is exactly 1. Remember, the probability of a random variable taking on a value between two values is given by the area under the PDF curve between those values.

2.2.1 Uniform Distribution

The uniform distribution describes a random variable for which any numerical value between the upper and lower limit is equally likely to occur. The PDF of a uniform distribution has the shape of a rectangle as shown in figure 2.4. This distribution is appropriate when limited information is available allowing an estimate of the minimum and maximum values, but not an estimate of the distribution shape. An example is a soils inventory that describes soil depth as between 3 and 10 feet. The uniform distribution would, of course, also be appropriate when the sample data suggest a uniformly distributed variable.

2.2.2 Triangular Distribution

The triangular distribution has the shape of a triangle that can be symmetrical or skewed in either direction (fig. 2.5). As with the uniform distribution, the triangular distribution would be used when relatively limited information is available; however, enough information should be available to estimate a most likely value as well as a minimum and maximum value. Note that the probability of a value occurring close to the minimum or maximum value of a triangular distribution is small, in contrast to a uniform distribution in which the probability of a value close to the minimum or maximum value occurring is the same as for any other value. Therefore, it is advisable when using a triangular distribution to extend the minimum and maximum values slightly beyond those you would specify for a uniform distribution.

2.2.3 Normal Distribution

The normal, or Gaussian, distribution has the familiar bell-shaped symmetry about the mean (fig. 2.6) and is defined by the mean and standard deviation. Of the total area under the normal curve, 68.26 percent occurs between the limits of the mean plus 1 standard deviation and the mean minus 1 standard deviation. This means that the probability of a normally distributed random variable having a value between the limits of the mean ± 1 standard deviation is 0.6826.

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathsf{E}[X] = \frac{a+b}{2}$$

$$\mathsf{Var}[X] = \left(\frac{b-a}{\sqrt{12}}\right)^2$$

USER INPUT: a and b

NOTATION: $U[a,b]$

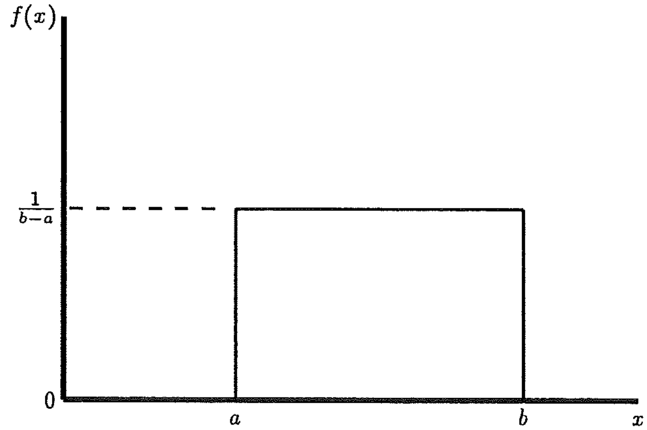


Figure 2.4—Uniform PDF.

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(c-a)(b-a)}, & \text{if } a \le x \le b; \\ \dfrac{2(c-x)}{(c-a)(c-b)}, & \text{if } b \le x \le c; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathsf{E}[X] = \frac{a+b+c}{3}$$

$$\mathsf{Var}[X] = \frac{a(a-b)+b(b-c)+c(c-a)}{18}$$

USER INPUT: a, b, and c

NOTATION: $T[a,b,c]$

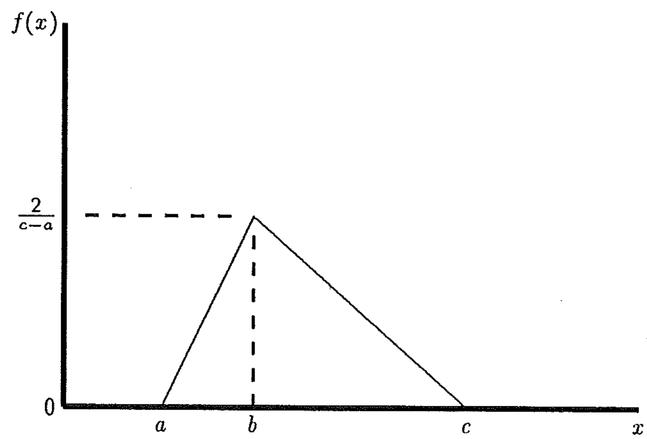


Figure 2.5—Triangular PDF.

Thus, when considering a sample data set, about 68 percent of the data points would be included in the interval defined by the mean ± 1 standard deviation if the random variable is normally distributed. Further, 95.84 percent of the total area under the curve is bound by the mean ± 2 standard deviations, and 99.72 percent by the mean ± 3 standard deviations.

Although the theoretical limits of the normal distribution are positive and negative infinity, LISA limits the distribution to ± 3.09 standard deviations (thereby sampling throughout 99.8 percent of the area under the normal PDF curve). These limiting values are indicated on the plot you obtain with the **Plot** option while viewing a data file (see part 2, section 3.10). Understanding these limits is helpful in estimating a realistic mean and standard deviation from limited information.

A good rule of thumb for estimating the standard deviation of a normally distributed variable is to divide the range by 4. For example, suppose that the forest soils inventory estimates that soil depths in a particular study area are in the range of 2 to 8 feet, and your past experience indicates that depths are likely normally distributed. A realistic mean and standard deviation would then be

$$f(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}, \qquad -\infty < x < \infty$$

$\mathrm{E}[X] = \mu$

$\mathrm{Var}[X] = \sigma^2$

USER INPUT: $\hat{\mu}$ and $\hat{\sigma}$
(approximately $\bar{x}$ and s for large samples)

NOTATION: $N[\bar{x}, s]$

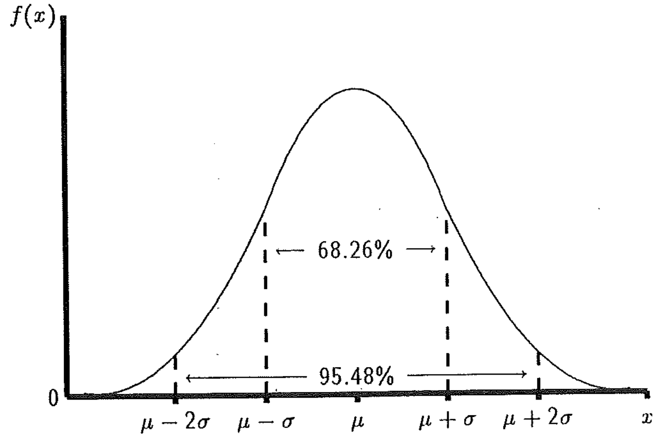


Figure 2.6—Normal PDF.

5 and 1.5 feet, respectively. LISA then will simulate values between 0.4 and 9.6 feet, with about 95 percent of the values between 2 and 8 feet.

If a standard deviation that is too large for a given mean is used, unrealistic endpoints for the normal distribution can result. For example, a normal distribution with a mean of 5 and a standard deviation of 3 will have limiting values of -4.3 and 14.3 (at the mean ± 3.09 standard deviations). Obviously, negative values for the physical factors in the infinite slope equation make no sense. To prevent simulation of negative values, LISA will check the value at the mean -3.09 standard deviations upon data entry, and if it is negative, LISA will display a warning message and wait for the user to enter values for the mean and standard deviation such that the value of the mean -3.09 standard deviations becomes positive.

2.2.4 Lognormal Distribution

The lognormal distribution is skewed to the right, indicating there is a relatively small probability of large values for the random variable. Figure 2.7 illustrates the general shape of the lognormal distribution.

If a random variable, X, is lognormally distributed, the logarithms of the values of X are normally distributed. By taking the logarithms of the values and computing the mean and standard deviation of these transformed values, one can use a standard normal distribution table to compute probabilities. One also can compute the mean and standard deviation of the logarithms of the values of X directly using the following formulas:

$$\hat{\mu}_l = \ln\bar{x} - \frac{\sigma_l^2}{2}$$

$$\hat{\sigma}_l = \sqrt{\ln\left[\frac{s^2}{x^2} + 1\right]}$$

where $\bar{x}$ and s are the mean and standard deviation of the actual data values, and $\hat{\mu}_l$ and $\hat{\sigma}_l$ are the estimated mean and standard deviation of the log-transformed variable, respectively. To simplify input, LISA users enter only the mean and standard deviation of the actual data values, $\bar{x}$ and s.

The shape of the lognormal distribution varies quite drastically with the coefficient of variation (c_v). If the c_v is less than about 0.08, the lognormal distribution is nearly symmetrical and looks like a normal distribution. As the c_v

$$f(x) = \begin{cases} \dfrac{e^{-0.5\left(\frac{\ln x - \mu_l}{\sigma_l}\right)^2}}{x\sigma_l\sqrt{2\pi}}, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{E}[X] = e^{\mu_l + 0.5\sigma_l^2}$$

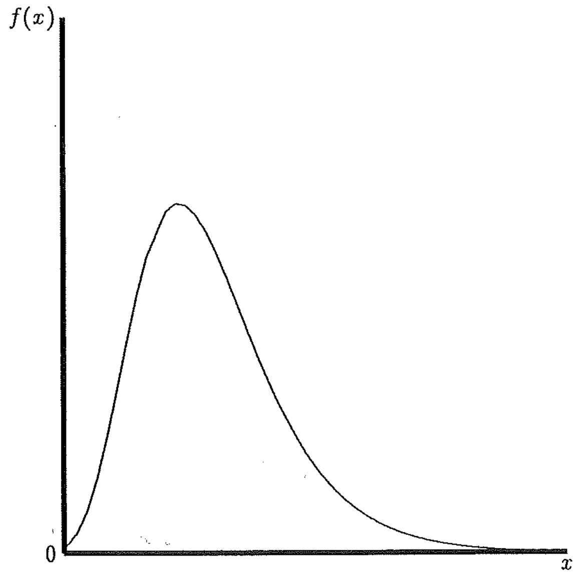
$$\mathrm{Var}[X] = \mathrm{E}[X]^2(e^{\sigma_l^2} - 1)$$

$$\mu_l = \ln(\mathrm{E}[X]) - \frac{\sigma_l^2}{2}$$

$$\sigma_l^2 = \ln\left\{\frac{\mathrm{Var}[X]}{\mathrm{E}[X]^2} + 1\right\}$$

USER INPUT: $\hat{\mu}$ and $\hat{\sigma}$

NOTATION: $L[\bar{x}, s]$



μ_l is the mean of the logarithms of the values of the random variable.
σ_l^2 is the variance of the logarithms of the values of the random variable.
$\hat{\mu}$ is approximately $\bar{x}$ of the values of the random variable for large samples.
$\hat{\sigma}$ is approximately s of the values of the random variable for large samples.

Figure 2.7—Lognormal PDF.

increases, the lognormal distribution becomes skewed more strongly to the right. The lognormal distribution is defined from zero to positive infinity, but LISA limits the simulation to values of the transformed mean (μ_l) ± 3.09 times the transformed standard deviation (σ_l). These values are shown on the distribution plot using the **Plot** option in LISA. The plotting option is helpful in selecting a mean and standard deviation that will give the desired shape and minimum and maximum values (see sections 3.10 and 3.11 in part 2).

2.2.5 Beta Distribution

The beta distribution requires four parameters to describe it—a minimum value (a), a maximum value (b), and two shape parameters (P and Q). The advantage of the beta distribution over some of the other distibution types is that the limits of the distribution are specified by the user, which eliminates the care required with the normal or lognormal distribution in the selection of a reasonable mean and standard deviation in order to obtain a realistic range of simulated values.

Also, the beta distribution can take on a wide variety of shapes; it can be skewed left, skewed right, or symmetrical, depending on the values of P and Q. In general, when P and Q are equal, the distribution is symmetrical; when P is greater than Q, the distribution is skewed left; and when P is less than Q, the distribution is skewed right. As the values of P or Q or both increase, the distribution becomes more peaked (greater kurtosis). Some of the possible shapes are shown in figure 2.8. Because the shape of the beta can be so variable, the **Plot** option in LISA is extremely useful in selecting appropriate P and Q values.

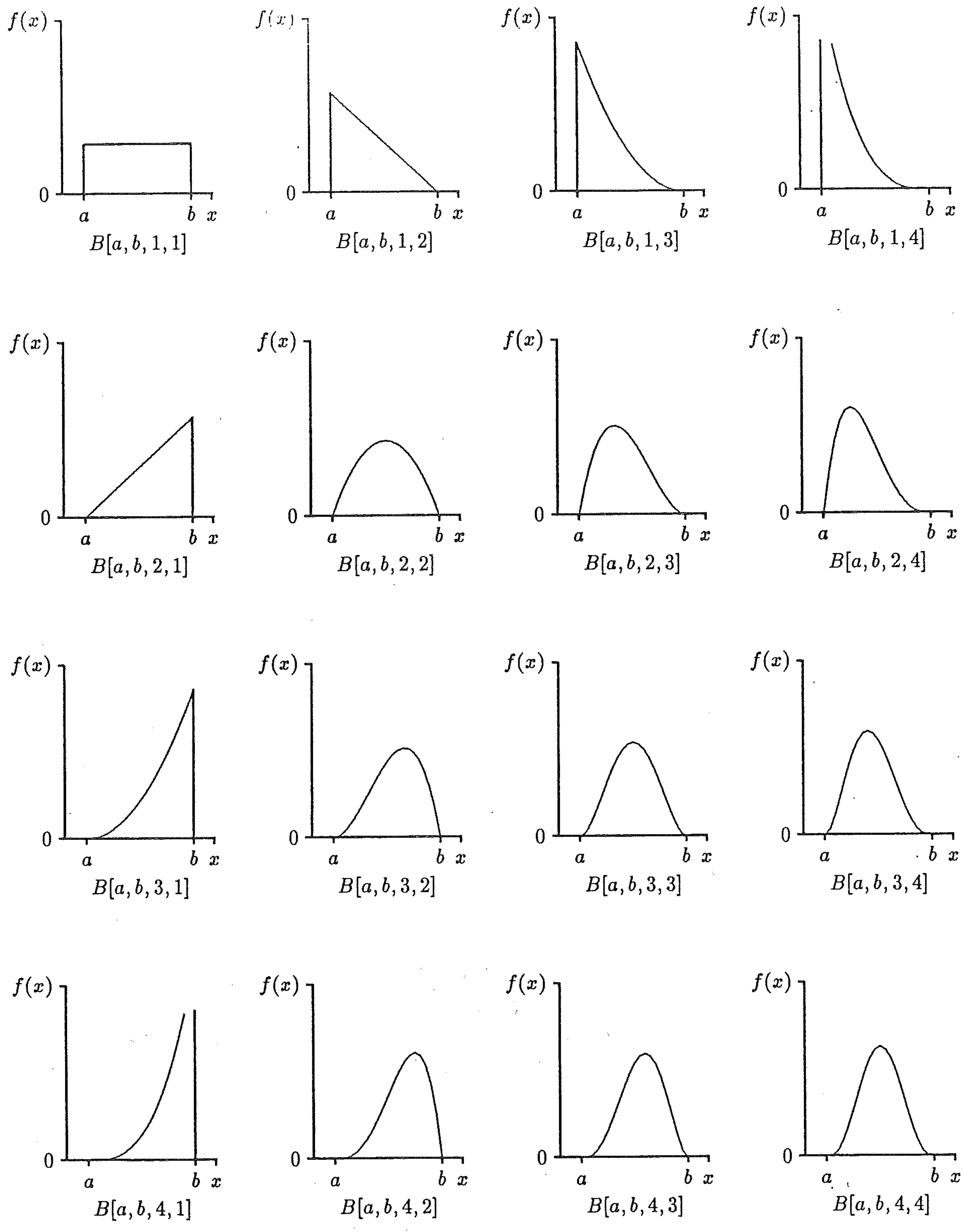


Figure 2.8—Beta PDF.

$$f(x) = \begin{cases} \frac{\Gamma(P+Q)}{\Gamma(P)\Gamma(Q)} \left[\frac{(x-a)^{P-1}(b-x)^{Q-1}}{(b-a)^{P+Q-1}} \right], & \text{if } a \le x \le b; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathrm{E}[X] = \frac{aQ + bP}{P + Q}$$

$$\mathrm{Var}[X] = \frac{(b-a)^2 PQ}{(P+Q)^2(P+Q+1)}$$

P and Q must be greater than 0.

Γ is the complete gamma function:

$$\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du = (a-1)\Gamma(a-1).$$

When a is a positive integer, then $\Gamma(a) = (a-1)!$.

USER INPUT: a,b,P,Q

NOTATION: $B[a, b, P, Q]$

Figure 2.8—(Con.)

P and Q also can be estimated from the sample mean ($\bar{x}$) and standard deviation (s) using the following equations:

$$\widehat{P} = \frac{\frac{(b-a)^2}{s^2} c - (c+1)^2}{(c+1)^3}$$

$$\widehat{Q} = \hat{P} c$$

where

$$c = \frac{b - \bar{x}}{\bar{x} - a},$$

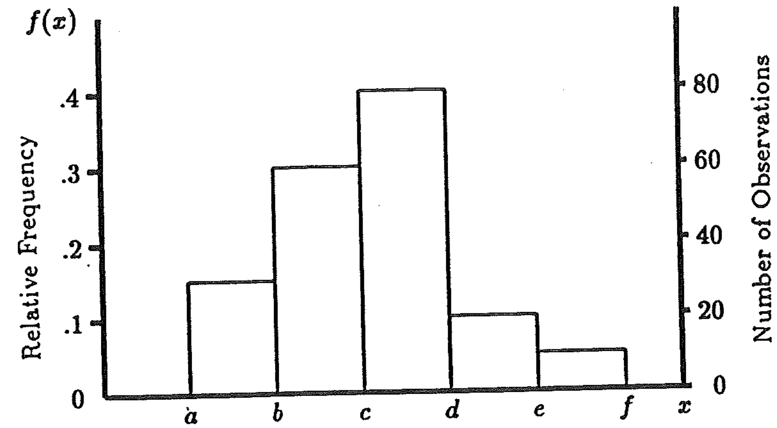
and a and b are the minimum and maximum values, respectively.

The disadvantage of using the beta distribution is that it requires approximately 20 to 30 times the computational time as do the other distributions. For example, it takes approximately 85 seconds to sample 1,000 values for the beta, while only 4 seconds for the other distributions on an 8.5 MHz machine with a math coprocessor; and 12 seconds for a beta while only 0.5 seconds for the others on a 20 MHz (80386) machine.

2.2.6 Relative-Frequency Histogram Distribution

A useful first step in selecting a PDF is to plot a histogram or relative-frequency histogram. This is done by grouping data into classes and then plotting a bar graph with the height of each bar equal either to the number of observations (to obtain a histogram), or to the relative frequency of observations (to obtain a relative-frequency histogram). The relative frequency is the number of observations in each class divided by the total number of observations. The histogram or relative-frequency histogram gives a good picture of the range and the distribution of data values. The relative-frequency histogram can be used directly in LISA, or the shape of the histogram or relative-frequency histogram might suggest another distribution that can be used to model the data.

Figure 2.9 shows an example histogram and relative-frequency histogram. Note that in LISA you enter the relative frequency expressed as a percentage.



$$f(x) = f_i = \frac{n_i}{n}$$

$$\mathrm{E}[X] = \frac{1}{n}\sum_{i=1}^{k} f_i x_i$$

$$\mathrm{Var}[X] = \frac{1}{n-1}\left[\sum_{i=1}^{k} f_i x_i^2 - \frac{(\sum^k f_i x_i)^2}{n}\right]$$

where:

k = number of classes
n = total number of observations
n_i = number of observations in the i^{th} class
f_i = frequency of observations in the i^{th} class
x_i = midpoint of the i^{th} class

Formulas apply only when classes are of equal widths

USER INPUT: Number of classes, values of class boundaries, percentage of observations in each class

NOTATION: $H[k, f_1, f_2, \ldots, f_k]$

Figure 2.9—A histogram PDF (right axis) and relative-frequency histogram PDF (left axis).

The relative frequency represents the probability of the random variable taking on a value in that class interval. Therefore, the percentages in all the classes must sum to 100 percent.

The appearance of the relative-frequency histogram can be affected significantly by the number and width of the class intervals used. Sturges (1926) suggests as a guide for selecting the number of classes of equal width

$$k = 1 + 3.3 \log_{10} n$$

where k is the number of classes and n is the number of data values. If too few classes are used, details of the shape of the data distribution will be lost. If too many are used, the histogram or relative-frequency histogram will appear erratic.

One comment on class width must be made. It can be convenient and is legitimate to use classes of unequal widths in the relative-frequency histogram. However, when unequal class widths are used, be aware that the relative-frequency histogram may give an incorrect picture of what the actual PDF looks like. This happens because the relative-frequency histogram is not a true PDF; that is, the *area* under the curve, computed as the sum of each class width times the frequency of observations in that class, does not, in general, equal 1. To obtain the true PDF, the frequency of observations in each class must be divided by the class width. This gives units on the y-axis of frequency per z, where z is in

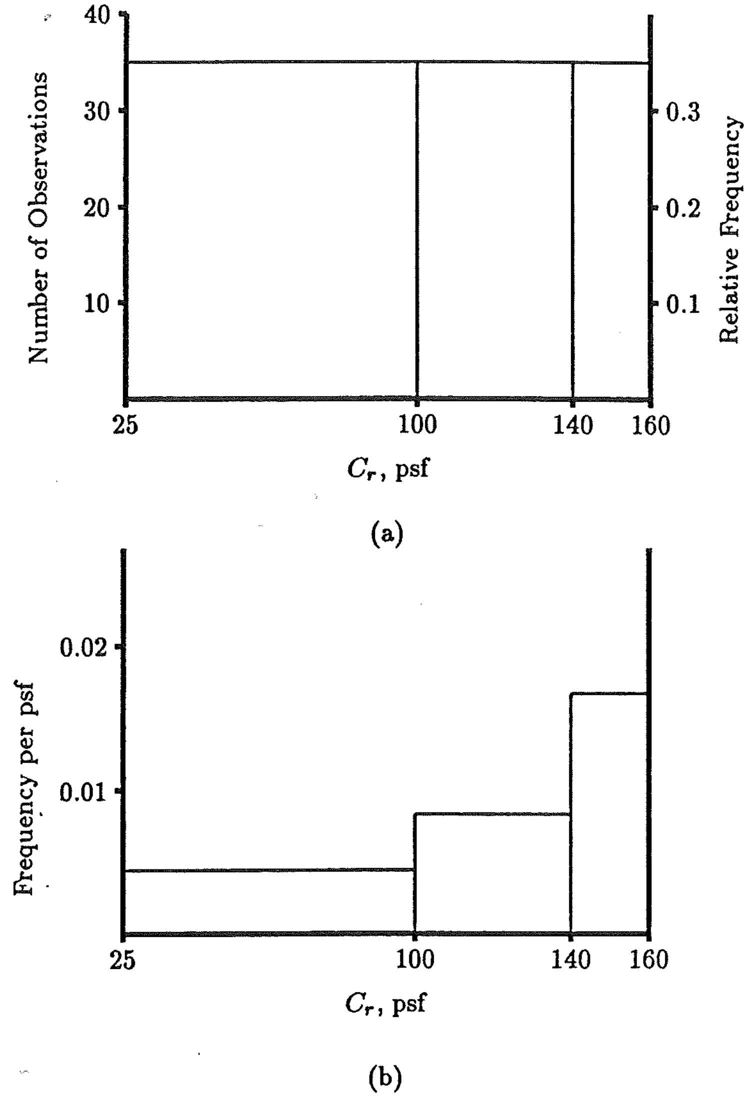


Figure 2.10—A histogram and relative-frequency histogram with unequal class widths (a) and the corresponding frequency density distribution (b).

the units of the random variable, and gives the area under the curve equal to 1. This true PDF is called a frequency density distribution.

Let's take a simple example. Figure 2.10a shows a histogram and relative-frequency histogram for 99 measurements of root strength (C_r) in which 33 measurements fall into each of three classes of unequal width. Notice that it looks like a uniform distribution. Figure 2.10b shows the frequency density distribution obtained by dividing 0.33, the relative frequency, by each class width. The shape of the distribution is drastically different, more like a triangular distribution. It is this true PDF that you see with the **Plot** option in LISA, and when you view the histogram of the simulated data after execution.

An example in which the use of unequal class widths is convenient is shown in figure 2.11. A soils inventory indicated that soil depths are predominantly between 24 and 48 inches with 15 percent of the landform having soils greater than 48 inches. Because the maximum soil depth is uncertain, several widths

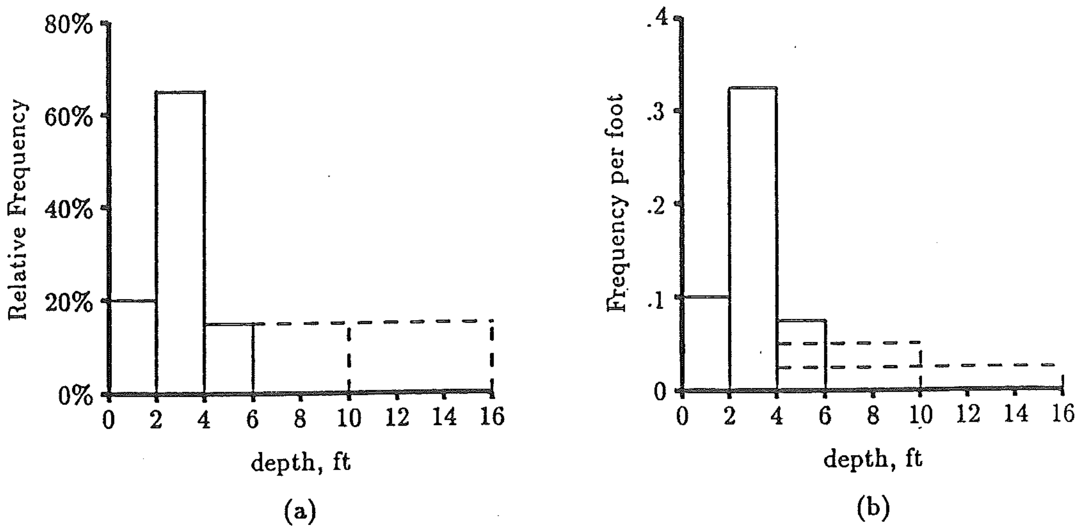


Figure 2.11—The relative-frequency histogram (a) and and the frequency density distribution (b) for a histogram at three possible maximum values.

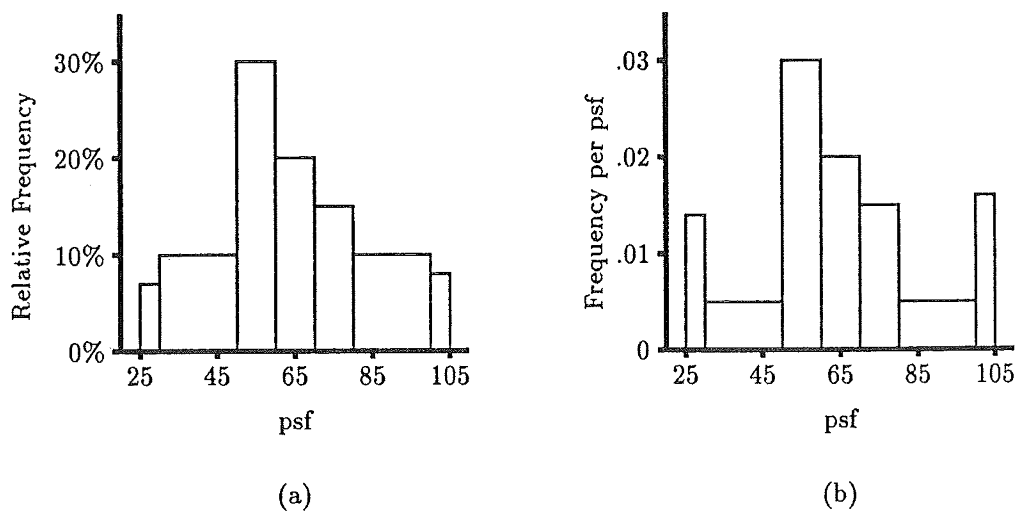


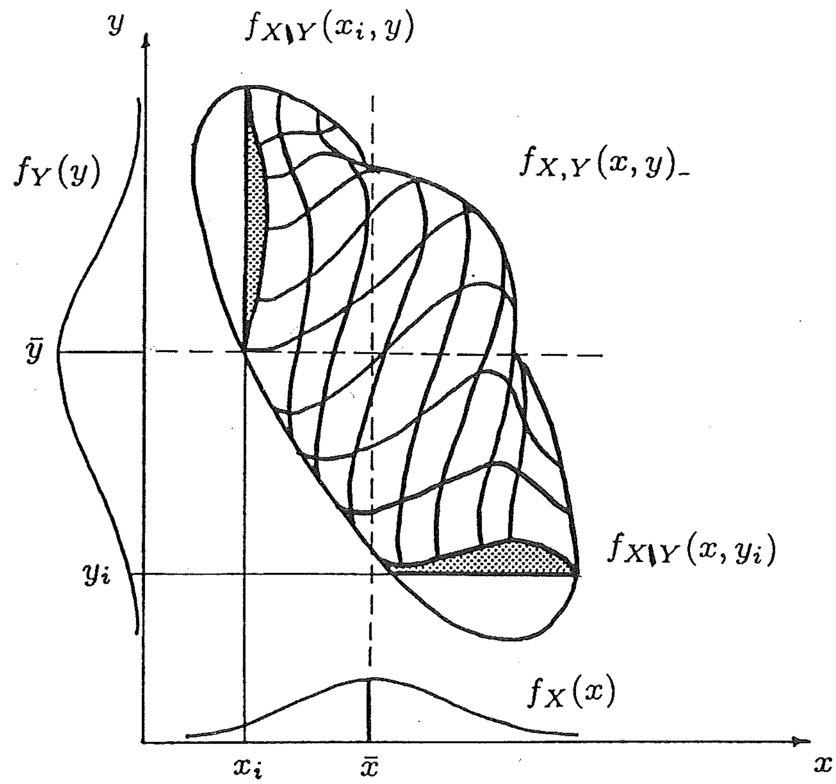
Figure 2.12—The relative-frequency histogram (a) and and the frequency density distribution (b) for a histogram of unequal class widths.

for the last class might be used to evaluate the sensitivity of the probability of failure to the maximum value. Figure 2.11b shows the effect of class width on class height in the frequency density distribution.

Another situation in which the difference between the relative-frequency histogram and the frequency density distribution appears is the case of narrow classes on the end of the histogram, as illustrated in figure 2.12. Tacking on narrow classes with small frequencies is an easy fix to make percentages sum to 100 percent. Just be aware that this can cause LISA to sample more values in those classes than may have been intended.

2.2.7 Bivariate Normal Distribution

The bivariate normal distribution is a joint PDF that can be used to model the linear correlation between C_s' and ϕ' (see section 4.2) and is available only for these two parameters. Figure 2.13 illustrates a bivariate normal PDF. The bivariate normal PDF is defined by specifying the normal marginal distributions



$x = \phi'$ $\quad y = C_s'$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 - 2r\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

$$-\infty \le x \le \infty, \quad -\infty \le y \le \infty$$

$\mathrm{E}[X] = \mu_X \qquad \mathrm{Var}[X] = \sigma_X^2 \qquad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right], \qquad -\infty \le x \le \infty$

$\mathrm{E}[Y] = \mu_Y \qquad \mathrm{Var}[Y] = \sigma_Y^2 \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right], \qquad -\infty \le y \le \infty$

USER INPUT: $\bar{x}, s_X, \bar{y}, s_Y, r_{X,Y}$

NOTATION: $BN[\bar{x}, s_X, \bar{y}, s_Y, r]$

Figure 2.13—A bivariate normal distribution.

for C_s' and ϕ' (that is, the means and standard deviations) and the correlation coefficient (r) between C_s' and ϕ'.